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ON ALMOST 1–1 EXTENSIONS

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ABSTRACT
We show that a broad class of extensions of measure preserving systems in the context of ergodic theory can be realized by topological models for which the extension is "almost one–one".

§1. It is well known that in a topological measure space one can have sets that are large topologically but small in the sense of the measure. In topological dynamics, when (X, τ) is a factor of (Y, τ) and the projection $\pi : Y \to X$ is one to one on a topologically large set (i.e. the complement of a set of first category), one calls (Y, τ) an *almost* 1–1 *extension* of (X, τ) and considers the two systems to be very closely related. Nonetheless, in view of our opening sentence it is possible that the measure theory of (Y, τ) will be quite different from the measure theory of (X, τ). We will prove here that indeed one can realize this possibility in an extreme way. Here is our main result:

THEOREM 1. *Let (X, τ) be a non-periodic minimal dynamical system, and let $\pi : Y \to X$ be an extension of (X, τ) with (Y, τ) topologically transitive and Y a compact metric space. Then there exists an almost* 1–1 *minimal extension of (X, τ), $(\bar{Y}, \bar{\tau})$, with $\bar{\pi} : \bar{Y} \to X$ and a Borel subset $Y_0 \subset Y$ with a Borel measurable map $\theta : Y_0 \to \bar{Y}$ satisfying* (1) $\theta\tau = \bar{\tau}\theta$, (2) $\bar{\pi}\theta = \pi$, (3) θ *is* 1–1 *on* Y_0, (4) $\mu(Y_0) = 1$ *for any τ invariant measure μ on Y.*

In words, one can find an almost 1–1 minimal extension of X such that the measure theoretic structure is as rich as that of an arbitrary topologically transitive extension of X. The next corollary answers a question raised by S. Glasner which provided the initial impetus for our work.

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COROLLARY 2. *Let* $(X, \mathscr{B}, \mu, T)$ *be an ergodic measure preserving transformation with non-trivial point spectrum described by* (G, ρ) *where* G *is a compact monothetic group* $\overline{\{\rho^n\}_{n\in\mathbf{Z}}} = G$. *Then there is an almost* 1–1 *minimal extension of* (G, ρ) (*i.e. a minimal almost automorphic system*), $(\tilde{Z}, \sigma)$, *and an invariant measure* ν *on* Z *such that* (Z, σ, ν) *is isomorphic to* $(X, \mathscr{B}, \mu, T)$.

This corollary shows that the measure theoretic character of almost automorphic systems is completely arbitrary. Another easy corollary of the main theorem is the following:

COROLLARY 3. *Any homomorphism of ergodic measure preserving transformations,*

$$\pi : (X_1, \mathscr{B}_1, \mu_1, T_1) \rightarrow (X_2, \mathscr{B}_2, \mu_2, T_2)$$

has a minimal model.

In the proof of this corollary one uses the Jewett–Krieger theorem to get a minimal (in fact strictly ergodic) model for the factor and then Theorem 1 enables one to lift to a minimal extension that captures $(X_1, \mathscr{B}_1, \mu_1, T_1)$. In [W] one can find a stronger version of the corollary with minimal replaced by strictly ergodic, but a rather more elaborate machinery is required to get it.

The nature of our proof of Theorem 1 is such that one can carry it out with minor modifications for any discrete group Γ instead of $\mathbf{Z}$. Using this we can construct an example of a minimal almost automorphic action of the free group that *has no invariant measure*. This answers a question raised several years ago by W. Veech.

§2. Proof of main theorem

Here is the strategy we will use in proving the main theorem. Fix some $y_0 \in Y$ with dense orbit and let $\pi y_0 = x_0$. We will construct a function $f: Y \rightarrow Y$ and then using f, $F: Y \rightarrow X \times Y^{\mathbf{Z}}$ will be defined by

$$F(y) = (\pi y, \{f(\tau^n y)\}_{n\in\mathbf{Z}}).$$

On $X \times Y^{\mathbf{Z}}$ we will define $\bar{\tau}$ by

$$\bar{\tau}(x, \{\eta_n\}_{n\in\mathbf{Z}}) = (\tau x, \{\eta_{n+1}\}_{n\in\mathbf{Z}})$$

so that F is equivariant. Finally $\bar{Y}$ will be the closure of $F(\tau^n y_0 : n \in \mathbf{Z})$ in

$X \times Y^{\mathbf{Z}}$ and $\bar{\pi}$ is the obvious projection onto the first coordinate. Soon Y_0 will be defined and then θ will be $F \mid Y_0$.

In constructing f it will be convenient to assume that X is totally disconnected. This is not a serious assumption inasmuch as one can always find a minimal 1-1 extension of X that is totally disconnected and then work over that extension. The main work will consist in finding a subset $J \subset \mathbf{Z}$ and a collection $\{U_j : j \in J\}$ of clopen sets in X, pairwise disjoint, with $\tau^j x_0 \in U_j$, and then f will be defined by

$$f(y) = \begin{cases} y & \text{if } \pi y \notin \bigcup_{j \in J} U_j, \\ \tau^j y_0 & \text{if } \pi y \in U_j. \end{cases}$$

Now the desired properties for $\bar{\pi}$, $\bar{Y}$ can be formulated in terms of the U_j's. In particular if $U = \bigcup_{j \in J} U_j$ includes the entire τ-orbit of x_0, then the $\bar{\pi}$ we defined above will be almost 1-1 since then $\bar{\pi}^{-1}\{x_0\}$ will consist of a single point. The minimality of $(\bar{Y}, \bar{\tau})$ is an easy formal consequence of the preceding property. The crucial property, of course, is the one that ensures that the orbit closure of $F(y_0)$ is large enough to accommodate all of the invariant measures that live on Y. This will be done by ensuring that

$$G_K = \{ y \in Y : \text{ there exists some } i \in \mathbf{Z} \text{ such that } d(f(\tau^k \tau^i y_0), f(\tau^k y)) < 1/K, \text{ all } |k| \leqq K \}$$

has full measure for any invariant measure on Y, and then Y_0 will be essentially $\bigcap G_K$. This last point will hopefully become clearer after we begin the construction itself. Now for the actual construction:

Step 1. Cover Y by finitely many open sets E_l, $1 \leqq l \leqq L$ so that for y_1, y_2 belonging to the same E_l,

$$d(\tau^k y_1, \tau^k y_2) < 1, \qquad \text{all } |k| \leqq 1.$$

Fix once and for all an ordering of $\mathbf{Z}$, say $\{0, +1, -1, +2, -2, \ldots\}$. By a simple inductive procedure define a finite set $J_1 \subset \mathbf{Z}$ with the properties:

(a) $0 \in J_1$, $|J_1| = L$,

(b) for each $1 \leqq l \leqq L$ there is a distinct $j_l \in J_1$ so that $\tau^{j_l} y_0 \in E_l$,

(c) $J_1 - 1$, J_1, $J_1 + 1$ are pairwise disjoint.

(Here $J_1 + a = \{j + a : j \in J_1\}$.) We are using, of course, the fact that y_0 has a dense orbit.

Denote $\hat{J}_1 = \bigcup_{|k| \leqq 1} (J_1 + k)$ and choose integers $0 = n_0, n_1, \ldots, n_9$ so that the sets $\{\hat{J}_1 + n_i\}_{i=0}^{9}$ are pairwise disjoint. Starting with some element of J_1, say 0, let U_0 be a clopen neighborhood of x_0 so small that

$$\{\tau^m U_0\}, \qquad m \in \bigcup_{i=0}^{9} (\hat{J}_1 + n_i)$$

are all pairwise disjoint. For the remaining $j \in J_1$ define

$$U_j = \tau^j U_0,$$

so that indeed $\tau^j x_0 \in U_j$ for all $j \in J_1$. Setting

$$V_1 = \bigcup_{|k| \leqq 1} \tau^k \Big(\bigcup_{j \in J_1} U_j \Big)$$

the auxiliary n_j's guarantee for us that the sets $V_1, \tau^{n_1} V_1, \dots, \tau^{n_9} V_1$ are disjoint and thus

$$\mu(\pi^{-1}(V_1)) \leqq \tfrac{1}{10}$$

for any τ-invariant measure μ. Finally put

$$f_1(y) = \begin{cases} y & \text{if } \pi(y) \notin \bigcup_{j \in J_1} U_j, \\ \tau^j y_0 & \text{if } \pi y \in U_j \text{ for some } j \in J_1. \end{cases}$$

If now $y \in E_l$, and $j_l \in J_1$ is such that $\tau^{j_l} y_0 \in E_l$, then as long as $y \notin \pi^{-1}(V_1)$

$$f_1(\tau^k y) = \tau^k y, \qquad |k| \leqq 1$$

and so

$$d(f_1(\tau^k y), f_1(\tau^k \tau^{j_l} y_0)) < 1 \qquad |k| \leqq 1.$$

The sets $U_j, j \in J_1$ will not be changed during the rest of the construction — although f_1 will be modified somewhat.

Since there is a new feature in the succeeding steps we will first carry out in detail step 2 — and then formulate the inductive step.

Step 2. The sets $\{\pi^{-1}(U_j), j \in J_1\}$, $Y \setminus \bigcup_{j \in J_1} \pi^{-1}(U_j)$ form a partition of Y into closed sets. Let δ_1 be the minimal distance between these sets and cover Y with open sets E_l^2, $1 \leqq l \leqq L_2$ so that for y_1, y_2 in the same E_l^2 we have

$$d(\tau^k y_1, \tau^k y_2) < \min\{\tfrac{1}{2}, \tfrac{1}{10}\delta_1\} \qquad \text{for all } |k| \leqq 2.$$

To define J_2, first construct a preliminary set I_2 with the properties:

(a) I_2 contains the first element of $\mathbf{Z}$ not in J_1, $|I_2| = L_2$;

(b) $I_2 \cap J_1 = \varnothing$;

(c) for each $1 \leqq l \leqq L_2$ there is a distinct $i \in I_2$ so that $\tau^i y_0 \in E_l^2$;

(d) the sets $\{I_2 + k : |k| \leqq 4\}$ are pairwise disjoint.

Now J_2 is defined to be all integers of the form $i + k$ with $i \in I_2$, $|k| \leqq 2$ such that

$$\tau^{i+k} x_0 \notin \bigcup_{j \in J_1} U_j.$$

Denoting $\hat{J}_2 = \bigcup_{|k| \leqq 2} J_2 + k$, find n_i's so that $\{\hat{J}_2 + n_i : 0 \leqq i < 100\}$ are pairwise disjoint. Starting with any $j_0 \in J_2$ take a clopen neighborhood U_{j_0} of $\tau^{j_0} x_0$ so small that the sets

$$\tau^{m-j_0} U_{j_0}, \qquad m \in \bigcup_{i=0}^{99} (\hat{J}_2 + n_i)$$

are pairwise disjoint. In addition we need that the sets $\{\tau^{m-j_0} U_{j_0} : m \in J_2\}$ are disjoint from the U_j's for $j \in J_1$. For $j \in J_2$ define

$$U_j = \tau^{j-j_0} U_{j_0}.$$

As before, setting

$$V_2 = \bigcup_{|k| \leqq 2} \tau^k \left(\bigcup_{j \in J_2} U_j \right)$$

we have arranged that

$$\mu(\pi^{-1}(V_2)) \leqq \tfrac{1}{100}$$

for any τ-invariant measure μ on Y.

Define

$$f_2(y) = \begin{cases} y & \text{if } \pi(y_0) \notin \bigcup_{j \in J_1 \cup J_2} U_j, \\ \tau^j y_0 & \text{if } \pi y \in U_j, \quad \text{for some } j \in J_1 \cup J_2. \end{cases}$$

If $y \notin \pi^{-1}(V_2)$ then $\tau^k y$ cannot be in $\pi^{-1}(\bigcup_{j \in J_2} U_j)$ for $|k| \leqq 2$. It is possible that $\tau^k y$ for some $|k| \leqq 2$ lands in $\pi^{-1}(U_j)$ for some $j \in J_1$ but in that case, if $i \in I_2$ was such that $\tau^i y_0$ and y are in the same E_l^2, we would have also $\tau^{k+i} y_0$ in the same $\pi^{-1}(U_j)$. It follows that we have

$$(*)\qquad d(f_2(\tau^{k+i}y_0), f_2(\tau^k y)) < \tfrac{1}{2} \qquad \text{for all } |k| \leqq 2$$

whenever $y \notin \pi^{-1}(V_2)$. Furthermore, we still keep for all $y \notin \pi^{-1}(V_1) \cup \pi^{-1}(V_2)$

$$(**)\qquad d(f_2(\tau^k y), f_2(\tau^{k+j}y_0)) < 1 \qquad \text{for } |k| \leqq 1$$

and suitably chosen $j \in J_1$. In spite of the fact that (*) is better than (**) and holds for a larger set we still must keep track of relations like (**) since we wish to make assertions about f which will be a pointwise limit of the f_n's.

It's time to formulate the data accumulated up to step M and indicate the inductive step. For each $m \leqq M$ we have

(i) sets of integers I_m, J_m $(I_1 = J_1)$, $\hat{J}_m = \bigcup_{|k| \leqq m} J_m + k$;

(ii) disjoint clopen sets $\{U_j\}$, $j \in \bigcup_1^M J_m$ and for each such j, $\tau^j x_0 \in U_j$;

(iii) $V_m = \bigcup_{|k| \leqq m} \tau^k(\bigcup_{j \in J_m} U_j)$, these $\tau^k U_j$'s are also disjoint and there are integers n_i, $0 \leqq i < 10^m$ such that $\{\tau^{n_i} V_m\}_{i=0}^{10^m-1}$ are pairwise disjoint (the dependence of the n_i's on m has been suppressed);

(iv) $\bigcup_1^M I_m$ includes the first M elements of $\mathbf{Z}$;

(v) for each $i \in I_m$, $|k| \leqq m$ either $\tau^{i+k}x_0 \in U_j$ for some $j \in J_1 \cup \cdots \cup J_{m-1}$ or $i + k \in J_m$;

(vi) the functions f_m defined by

$$f_m(y) = \begin{cases} y & \text{if } \pi(y) \notin \bigcup_{n=1}^{m} \bigcup_{j \in J_n} U_j \\ \tau^j y_0 & \text{if } \pi(y) \in U_j, \quad \text{for some } j \in \bigcup_{n=1}^{m} J_n \end{cases}$$

have the following property:

For any $K \leqq m$ and any $y \notin \pi^{-1}(V_K) \cup \cdots \cup \pi^{-1}(V_m)$ there is some $i \in I_K$ so that for all $|k| \leqq K$

$$(*)\qquad d(f_m(\tau^k y), f_m(\tau^{k+i}y_0)) < 1/K.$$

In fact this last property follows from the following more detailed information: for $y \notin \pi^{-1}(V_K) \cup \pi^{-1}(V_{K+1}) \cup \cdots \cup \pi^{-1}(V_m)$ there is some $i \in I_K$ such that either $f_m(\tau^k \tau^i y_0) = \tau^{k+i}y_0$ and $f_m(\tau^k y) = \tau^k y$ and (*) follows because $\tau^{k+i}y_0$ was sufficiently close to $\tau^k y$; or $\pi(\tau^{k+i}y_0) \in U_j$ for some $j \in \bigcup_1^{K-1} J_n$ and then also $\tau^k y \in U_j$ and (*) follows because f_m of both points equals $\tau^j y_0$.

Step $M+1$. (a) Determine a δ_{M+1} to be less than $1/(M+1)$ and *also* less than the minimum distance between the U_j's, $j \in \bigcup_1^M J_m$ and between them and the complementary set. Then let $\{E_l^{M+1}\}$, $1 \leqq l \leqq L_{M+1}$, be a finite open cover of Y so that for any u, v in the same open set of this cover

$$d(\tau^k u, \tau^k v) < \tfrac{1}{10}\delta_{M+1} \qquad \text{all } |k| \leqq M+1.$$

(b) Next define a finite set $I_{M+1} \subset \mathbf{Z}$ with the properties:

(1) I_{M+1} contains the first element of $\mathbf{Z}$ not in $\bigcup_1^M J_m$

(2) $|I_{M+1}| = L_{M+1}$,

(3) $I_{M+1} \cap (\bigcup_1^M J_m) = \varnothing$,

(4) for each $1 \leqq l \leqq L_{M+1}$ there is a distinct $i \in I_{M+1}$ with $\tau^i y_0 \in E_l^{M+1}$,

(5) the sets $\{I_{M+1} + k : |k| \leqq 2(M+1)\}$ are pairwise disjoint.

This is easily done by an inductive procedure using the fact that the orbit $\{\tau^n y_0\}_{n \in \mathbf{Z}}$ is dense in Y.

(c) Define J_{M+1} as the set of all integers of the form $i+k$, $i \in I_{M+1}$, $|k| \leqq M+1$ such that

$$\tau^{i+k} x_0 \notin \bigcup_{m=1}^{M} \bigcup_{j \in J_m} U_j.$$

In addition set

$$\hat{J}_{M+1} = \bigcup_{|k| \leqq M+1} (J_{M+1} + k).$$

Note that the sets $J_{M+1} + k$, $|k| \leqq M+1$ are pairwise disjoint.

(d) Choose integers n_i, $0 \leqq i < 10^{M+1}$ so that the sets $\{\hat{J}_{M+1} + n_i\}_{i=0}^{10^{M+1}}$ are pairwise disjoint. Now, starting with any fixed $j_0 \in J_{M+1}$ find a clopen neighborhood of $\tau^{j_0} x_0$, U_{j_0}, so small that:

the sets $\tau^{m-j_0} U_{j_0}$ are pairwise disjoint for all

$$m \in \bigcup_{0 \leqq i < 10^{m+1}} (\hat{J}_{m+1} + n_i)$$

and also disjoint from the previous U_j's.

(e) Set

$$V_{m+1} = \bigcup_{j \in J_{m+1}} \bigcup_{|k| \leqq m+1} \tau^k U_j.$$

By (c), (d) the sets $\tau^{n_i} V_{m+1}$, $0 \leqq i < 10^{m+1}$, are pairwise disjoint and thus for any τ-invariant measure μ

$$\mu(V_{m+1}) \leqq 10^{-(M+1)}.$$

Finally define

$$f_{M+1} = \begin{cases} y & \text{if } \pi(y) \notin \bigcup_{m=1}^{M+1} \bigcup_{j \in J_m} U_j, \\ \tau^j y_0 & \text{if } \pi(y) \in U_j, \quad \text{for some } j \in \bigcup_{1}^{M+1} J_m. \end{cases}$$

It is completely straightforward to check that with these definitions the properties (i)–(vi) listed above continue to hold for $M+1$. Doing this for all M we finally define

$$f = \lim f_n.$$

From the definition of the f_n's one sees that this limit is given by the formula

$$f(y) = \begin{cases} y & \text{if } \pi(y) \notin \bigcup_{m=1}^{\infty} \bigcup_{j \in J_m} U_j, \\ \tau^j y & \text{if } \pi(y) \in U_j, \quad \text{for } j \in \bigcup_{1}^{\infty} J_m = J. \end{cases}$$

Note that (b)-(1) ensures that $\bigcup_{j \in J} U_j$ contains the full orbit of x_0. Also by property (vi) we get for each y not in $\bigcup_{m \geqq K} \pi^{-1}(V_m)$ an integer i_K such that

$$(**) \qquad d(f(\tau^k y), f(\tau^{k+i_K} y_0)) < 1/K \qquad \text{all } |k| \leqq K.$$

Furthermore, for any invariant measure μ

$$\mu\left(\bigcup_{m \geqq k} \pi^{-1}(V_m)\right) \leqq 1/10^{k-1}.$$

Thus for any y not in the μ-null set

$$\bigcap_k \left(\bigcup_{m \geqq k} \pi^{-1}(V_m)\right)$$

there is a sequence of indices i_k, $k \geqq k(y)$ with (**) valid.

Recall now the definition of F,

$$F(y) = (\pi(y); \{f(\tau^n y)\}_{n \in \mathbf{Z}}) \in X \times Y^{\mathbf{Z}}.$$

The only place where F can fail to be 1–1 is on the fibers $\pi^{-1}(\{x\})$. As long as for some n, $\tau^n(\pi(y))$ avoids $U = \bigcup_{j \in J} U_j$, F will be 1–1 on the whole fiber to which y belongs. By our construction this set has full measure for any invariant measure μ. Together with the property above we get that F maps a set of full μ-measure in Y in a 1–1 fashion into *a subset of the closure of* $F\{\tau^n y_0\}$ *in* $X \times Y^{\mathbf{Z}}$.

We shall now check in detail the fact that $\bar{Y} = \bar{F}\{\tau^n y_0 n \in \mathbf{Z}\}$ is a minimal almost 1–1 extension of X. Indeed since U is an open set containing the orbit of x_0, and for any $x \in U$, f is constant on $\pi^{-1}(\{x\})$, for any N we can find a neighborhood W of x_0 such that $\tau^n W \subset U$, $|n| \leqq N$, and then for all $y \in \pi^{-1}(W)$, $f(\tau^n y)$ depends only on $\tau^n \pi(y)$ for all $|n| \leqq N$ so that the diameter of

$$F(\pi^{-1}(W)) = \bar{\pi}^{-1}(w) \cap Y^{\mathbf{Z}}$$

will tend to zero as $N \to \infty$ and W (which depends of course on N) decreases to x_0.

To see that $(\bar{Y}, \bar{\tau})$ is *minimal* let $\mathbf{Z} \subset \bar{Y}$ be a non-empty $\bar{\tau}$-invariant closed set. Since X is minimal and $\bar{\pi}(\mathbf{Z})$ in X will be τ-invariant we have $\bar{\pi}(\mathbf{Z}) = X$. Since $\bar{\pi}$ is 1–1 over the point x_0 it follows that $\mathbf{Z}$ includes $F(y_0)$ and then $\mathbf{Z} = \bar{Y}$ by the definition of $\bar{Y}$ as the orbit closure of $F(y_1)$.

REMARK. It is worth mentioning that we did not use the fact that (X, τ) was minimal in the construction of $\bar{Y}$. It played a role only in showing that $(\bar{Y}, \bar{\tau})$ was minimal. Thus we could formulate a more general result in which (X, τ) is taken to be topologically transitive. Since we have no immediate application of such a result we refrain from entering into any more details.

§3. The case of a general discrete group

In this section Γ will denote some fixed infinite discrete group. We suppose that Γ acts transitively on Y (say that Γy_0 is dense in Y) and $\pi : Y \to X$ is an equivariant map such that the action of Γ on X is minimal and X is totally disconnected. In the construction of f in §2, we made no use of the specific structure of $\mathbf{Z}$, and so we could have carried it out for any group Γ. It will suffice to describe the elements of the construction; verifying that it is feasible as well as checking that it will give the desired result may be left as an exercise.

The role of the sets $\{k : |k| \leqq K\}$ will be taken over by sets B_n, finite symmetric sets containing the identity such that:

$$B_n^2 \subset B_{n+1} \quad \text{all } n; \qquad \bigcup_1^\infty B_n = \Gamma.$$

Sets $I_m, J_m \subset \Gamma$ will be constructed so that:

$$\bigcup_1^\infty I_m = \Gamma,$$

for each $\gamma \in J_m$, there is a clopen set U_γ containing γx_0, such that for fixed m, and $\gamma_1, \gamma_2 \in J_m$,

$$U_{\gamma_2} = \gamma_2 \gamma_1^{-1} U_{\gamma_1}.$$

Also the U_γ's, $\gamma \in J = \bigcup_1^\infty J_m$, are disjoint.

For any $\gamma \in I_m$, $\alpha \in B_m$ either $\alpha\gamma \in J_m$ or

$$\alpha\gamma x_0 \in U_\gamma, \qquad \gamma \in J_i \quad \text{for some } i < m.$$

The sets $\{\alpha U_\gamma : \alpha B_m, \gamma \in J_m\}$ are disjoint and furthermore

$$V_m = \bigcup_{\alpha \in B_m} \bigcup_{\gamma \in J_m} \alpha U_\gamma$$

have enough disjoint translates so as to force their measure to be less than 10^{-m} for any Γ-invariant measure on X.

The function $f: Y \to Y$ will be defined as before as a limit of f_m's where

$$f_m(y) = \begin{cases} y & \text{if } \pi(y) \notin \bigcup_{i \leqq m} \bigcup_{\gamma \in J_i} U_\gamma, \\ \gamma y_0 & \text{if } \pi(y) \in U_\gamma, \quad \text{for } \gamma \in \bigcup_{i \geqq m} J_i. \end{cases}$$

The crucial property is this: if $y \notin \pi^{-1}(V_K) \cup \cdots \cup \pi^{-1}(V_m)$ then there is some $\beta \in I_K$ such that βy_0 is very close to y and for all $\alpha \in B_K$ either

$$f_m(\alpha\beta y_0) = \alpha\beta y_0 \quad \text{and} \quad f_m(\alpha y) = \alpha y$$

or $\alpha\beta y_0 \in \pi^{-1}(U_\gamma)$ for some $\gamma \in J_i$, $i < K$ and then, since y was close enough to βy_0, also $\alpha y \in \pi^{-1}(U_\gamma)$ for the same γ. This enables us to prove:

Theorem 4. *Let Γ be an infinite group acting minimally on a totally disconnected space X, $\pi: Y \to X$ an extension of X on which Γ acts topologically* *transitively. Then there exists an almost* 1–1 *minimal extension* $(\bar{Y}, \Gamma)$ *of* X, $\bar{\pi}: \bar{Y} \to X$ *and a Borel subset* $Y_0 \subset Y$ *with a map* $\theta: Y_0 \to \bar{Y}$ *satisfying*:

(1) $\theta\gamma = \gamma\theta$ *for all* $\gamma \in \Gamma$; $\bar{\pi}\theta = \pi$,

(2) θ *is* 1–1 *on* Y_0,

(3) $\mu(Y_0) = 1$ *for any* Γ*-invariant measure* μ *on* Y.

In fact the set Y_0 *will be of the form* $\pi^{-1}(X_0)$ *for some set* $X_0 \subset X$ *and* (3) *may be replaced by* $\mu(X_0) = 1$ *for any* Γ*-invariant measure* μ *on* X.

The last remark means that if $(\bar{Y}, \Gamma)$ has an invariant measure $\bar{\mu}$ that projects onto μ on X, then θ^{-1} will take $\bar{\mu}$ onto a Γ-invariant measure on Y and give in fact an isomorphism between $(\bar{Y}, \Gamma, \bar{\mu})$ and $(Y, \Gamma, \theta^{-1}\bar{\mu})$. We apply this remark in the next section.

§4. Almost automorphic actions

Let Γ act on a compact metric space X. The action is said to be *almost automorphic* if for some point $x_0 \in X$, (i) the orbit Γx_0 is dense and (ii) whenever $\gamma_n x_0 \to x$ for some $\{\gamma_n\} \subset \Gamma$ and $x \in X$, we also have $\gamma_n^{-1} x \to x_0$. Veech has shown [V] that the almost automorphic actions of a group Γ are exactly the almost 1–1 extensions of the minimal equicontinuous actions of Γ. Now, equicontinuous actions for any group always possess invariant measures. Veech has raised the question as to whether there necessarily exist invariant measures for almost automorphic actions of arbitrary (non-amenable) discrete groups Γ. We shall show that this is not the case.

Let Γ be the free group on r generators, $a_1, a_2, \ldots, a_r$. Let Ω be the set of one-sided infinite sequences with entries a_i or a_i^{-1} subject to the condition that a_i and a_i^{-1} never appear consecutively. We can regard Γ as the set of finite words of the same sort, and we can define an action $\Gamma \times \Omega \to \Omega$ by letting $\gamma(\omega)$ be the infinite sequence obtained by juxtaposing γ and ω and cancelling any consecutive occurrence of a generator and its inverse. Ω is a compact metrizable space in a natural way and it is easily checked that Ω is a "boundary" of Γ in the sense of [F], so that for any probability measure ν on Ω, there exists a sequence $\{\gamma_n\}$ in Γ with $\gamma_n\nu \to$ a point measure.

Now let Z be the profinite closure of Γ, so that Z is a compact group with Γ as a dense subgroup. Γ acts on Z by left multiplication, and clearly Haar measure on Z, m_Z, is invariant for this action.

LEMMA. *The action of* Γ *on* $Z \times \Omega$ *is minimal.*

PROOF. Let $A \subset Z \times \Omega$ be a closed Γ invariant set. Since Γ is dense in

Z, the projection $A \to Z$ is onto. Let $\tilde{\lambda}$ be a probability measure on A mapping onto m_z under this projection, and let λ be the projection of $\tilde{\lambda}$ into Ω. For some $\{\gamma_n\}$ we have $\gamma_n \lambda \to \delta_{\omega_0}$ for a point $\omega_0 \in \Omega$. Pass to a subsequence so that $\gamma_n \lambda$ converges, say to ν. Then ν is a probability measure on A mapping onto δ_{ω_0} under the projection $Z \times \Omega \to \Omega$. On the other hand $\gamma_n \tilde{\lambda}$ maps to m_Z for $Z \times \Omega \to Z$, and so maps onto m_Z. Hence $\nu = m_Z \times \delta_{\omega_0}$. But if this measure sits in A, so does $m_Z \times \delta_\omega$ for a dense set of ω. This proves $A = Z \times \Omega$. □

We now apply Theorem 4 to $Y = Z \times \Omega$ and $X = Z$. Γ acts minimally on Y and so it certainly acts topologically transitively. Let $\bar{Y}$ be the almost 1–1 extension of Z whose existence is guaranteed by Theorem 4. $(\bar{Y}, \Gamma)$ is an almost automorphic action. Suppose this action has an invariant measure $\bar{\mu}$. $\bar{\mu}$ necessarily projects onto m_Z, the unique invariant measure for (Z, Γ). We conclude that Γ leaves some measure invariant on $Y = Z \times \Omega$. But this is absurd since Γ has no invariant measure on Ω. This answers Veech's question.

References

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